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# TOPOLOGICAL EQUIVALENCE OF DISCONTINUOUS NORMS*

BY

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ABSTRACT

We show that for every $p > 0$ there is an autohomeomorphism $h$ of the countable infinite product of lines $\mathbf{R}^{\mathbf{N}}$ such that for every $r > 0$, $h$ maps the Hilbert cube $[-r, r]^{\mathbf{N}}$ precisely onto the "elliptic cube" $\{x \in \mathbf{R}^{\mathbf{N}} : \sum_{i=1}^{\infty} |x_i|^p \leq r^p\}$. This means that the supremum norm and, for instance, the Hilbert norm ($p = 2$) are topologically indistinguishable as functions on $\mathbf{R}^{\mathbf{N}}$. The result is obtained by means of the Bing Shrinking Criterion.

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## 1. Introduction

If $X$ is a (real) topological vector space endowed with the continuous norms $\|\cdot\|$ and $|\cdot|$, respectively, then there is a norm preserving homeomorphism $f\colon (X,\|\cdot\|) \to (X,|\cdot|)$ defined by $f(\mathbf{0}) = \mathbf{0}$ and

$$(*) \qquad f(x) = \frac{\|x\|}{|x|}x$$

if $x \neq \mathbf{0}$. Observe that such a homeomorphism is in general not linear. Consider, for example, $\mathbf{R}^2$ endowed with the Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ and the max norm $|x| = \max\{x_1, x_2\}$. So a norm preserving homeomorphism sends the unit ball $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ onto the unit brick $[-1,1]^2$ and consequently changes the shape of a geometric object considerably.

These considerations for continuous norms are not very interesting and the question naturally arises whether something can be said in the case of discontinuous norms. All norms on finite-dimensional vector spaces are continuous, so the question only makes sense within the framework of infinite-dimensional spaces. If $X$ is an infinite-dimensional vector space, then it can be endowed with several discontinuous norms. Consider, for example, the case of $\ell^p$ and $\ell^q$, where $p < q$. Then as vector spaces, $\ell^p$ and $\ell^q$ are isomorphic (they have Hamel bases of the same cardinality) and so under this equivalence the norm on $\ell^q$ defines a norm on $\ell^p$ which is badly discontinuous of course.

In this paper we are interested in the classical sequence spaces

$$\ell^p = \begin{cases} \{(x_n)_n \in \mathbf{R}^{\mathbf{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}, & \text{for } 0 < p < \infty, \\ \{(x_n)_n \in \mathbf{R}^{\mathbf{N}} : \sup\{|x_n| : n \in \mathbf{N}\} < \infty\}, & \text{for } p = \infty, \end{cases}$$

endowed with their $p$-norms:

$$(**) \qquad |x|_p = \begin{cases} (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}, & \text{for } p < \infty, \\ \sup\{|x_n| : n \in \mathbf{N}\}, & \text{for } p = \infty. \end{cases}$$

We consider two vector space topologies on $\ell^p$: the natural norm topology and the topology of pointwise convergence. Observe that $\ell^p$ for $p \in (1,\infty)$ has the property that the weak topology on one the unit balls coincides with the topology of pointwise convergence. So the topology of pointwise convergence is natural if one wishes to deal with a metrizable topology in which balls are compact. This explains our interest in the topology of pointwise convergence on $\ell^\infty$. Its unit ball is a classical object: it is the familiar Hilbert cube $Q = \prod_{n=1}^{\infty}[-1,1]_n$ with the product topology. And its norm when restricted to nonnegative sequences in $Q$ is simply the supremum of the sequence. The Hilbert cube with the sup norm surfaced at other places in the literature. See, e.g., [6] where it is shown that it is a universal space for compact spaces with a lower semi-continuous function to the unit interval $[0, 1]$.

It is well known that the $p$-norms are lower semi-continuous but not continuous in the topology of pointwise convergence. In §2 of the present paper we shall prove that if $p, q < \infty$, then there exists a natural norm preserving homeomorphism $\ell^q \to \ell^p$. Observe that a formula such as the one in $(*)$ does not work since the norms involved are not continuous. Nonetheless, we prove that such a homeomorphism can be defined via another simple formula. The main part of the paper is devoted to proving that there also exists for every $p < \infty$ a norm preserving homeomorphism $\ell^\infty \to \ell^p$. A moment's reflection shows that for this quite a lot has to be achieved. For example, if $p = 2$ then the homeomorphism maps for every $\varepsilon > 0$ the "brick" $\prod_{n=1}^{\infty}[-\varepsilon, \varepsilon]_n$ onto the ellipsoid $\{x \in s : \sum_{i=1}^{\infty} x_i^2 \leq \varepsilon^2\}$. No simple formula achieves this. Pushing one specific brick onto one specific ellipsoid is no problem, but the need to do this for all bricks and all ellipsoids simultaneously creates tremendous problems. In fact, our homeomorphisms are constructed by means of Bing shrinking, a powerful tool from geometric topology.

It will be convenient to think of all the spaces $\ell^p$ for $p \in (0, \infty]$ as being situated in the ambient space $s$, the countable infinite product of real lines endowed with the topology of pointwise convergence. The norms in $(**)$ can be extended to norms on $s$ if one allows that the length of a vector can be $\infty$. The vectors in the complement of $\ell^p$ then simply have $p$-norm $\infty$ and on $\ell^p$ itself everything stays as before. All our homeomorphisms are ambient homeomorphisms, i.e., they are defined on all of $s$ and are norm preserving with respect to the extended norm.

We finish this introduction by making some remarks.

First, infinite-dimensional topology aims among other things at proving the homeomorphy of certain infinite-dimensional geometric objects that are not homeomorphic via a homeomorphism preserving their respective geometric structures. A good example of this is the celebrated Anderson Theorem [1] about the homeomorphy of separable real Hilbert space $\ell^2$ and $s$. Clearly, $\ell^2$ and $s$ are not linearly homeomorphic. Our results fit in very well in the program of infinite-dimensional topology. The homeomorphisms that we construct cannot preserve linearity, but do preserve the length of vectors. They show that at least part of the geometric structures under consideration can be preserved. As far as we know, our paper is the first attempt in infinite-dimensional topology to partly close the gap between homeomorphisms without additional properties and linear homeomorphisms.

Second, the method of absorbing sets in infinite-dimensional topology does not work in our situation. This is so because this method allows one to push a family of flexible *dense* sets in place (see, e.g., Dijkstra and Mogilski [2]) while we have to push uncountably many *compact* (= small) sets in place.

Third, a famous theorem of Keller [4] asserts that a Keller cube, i.e., an infinite-dimensional compact convex set in a separable metrizable locally convex vector space, is homeomorphic to $Q$. If $B$ is a reflexive separable Banach space, then its unit ball in the weak topology is a Keller cube. So one may ask whether Keller cubes can be homeomorphic via norm preserving homeomorphisms. Our results imply that for certain classical examples this is indeed the case. For consider $\ell^p$ for $p \in (1, \infty)$. Its unit ball in the weak topology is the subspace $B_p = \{x \in s : |x|_p \le 1\}$ of $s$. Our homeomorphisms show that those sets are all homeomorphic via norm preserving homeomorphisms. In fact, they are all norm-homeomorphic to the simplest Keller cube endowed with the simplest norm: the Hilbert cube with the sup-norm.

## 2. $p$-norms

Let $s$ stand for the countable product of lines $\mathbf{R}^{\mathbf{N}}$. We define the following familiar norms from $s$ to $[0, \infty]$: if $p \in (0, \infty]$ and $x = (x_1, x_2, \ldots)$ then

$$|x|_p = \begin{cases} (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}, & \text{for } p < \infty, \\ \sup\{|x_n| : n \in \mathbf{N}\}, & \text{for } p = \infty. \end{cases}$$

It is well known that these norms are lower semi-continuous but not continuous on $s$. A function $f\colon X \to [0, \infty]$ is called lower semi-continuous if the preimage of every interval $(r, \infty]$ is open. If $r \in \mathbf{R}$, then we define $\operatorname{sgn}(r)$ by $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(r) = r/|r|$ for $r \ne 0$. We call a function $f\colon s \to s$ sign preserving if for each $x, y \in s$ with $f(x) = y$ and each $n \in \mathbf{N}$ we have $y_n = 0$ or $\operatorname{sgn}(y_n) = \operatorname{sgn}(x_n)$. If $x = (x_1, x_2, \ldots) \in s$, then $\pi_n(x) = x_n$ for $n \in \mathbf{N}$. For $n \in \omega$ let $\xi_n\colon s \to s$ be the projection $\xi_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. So $\xi_0$ maps the whole space onto the origin. Note that $|\xi_n(x)|_p \le |\xi_{n+1}(x)|_p$ for every $x \in s$, $p \in (0, \infty]$, and $n \in \omega$.

For each $p \in (0, \infty)$ and $q \in (0, \infty]$ let $H^p_q\colon s \to s$ be a sign preserving function with the property that for each $x, y \in s$ and $n \in \mathbf{N}$ with $H^p_q(x) = y$ we have $|\xi_n(y)|_p = |\xi_n(x)|_q$. We show that every $H^p_q$ is uniquely determined. Let $x \in s$, $n \in \mathbf{N}$, and put $y = H^p_q(x)$. Consider

$$|y_n|^p = |\xi_n(y)|^p_p - |\xi_{n-1}(y)|^p_p = |\xi_n(x)|^p_q - |\xi_{n-1}(x)|^p_q.$$

Consequently, we have

$$|y_n| = \sqrt[p]{|\xi_n(x)|^p_q - |\xi_{n-1}(x)|^p_q}$$

and, since $H_q^p$ is sign preserving,

$$y_n = \operatorname{sgn}(x_n) \sqrt[p]{|\xi_n(x)|_q^p - |\xi_{n-1}(x)|_q^p}.$$

The latter formula can be used as a definition for $H_q^p$, establishing uniqueness. This formula also has the following obvious consequence:

PROPOSITION 2.1: *If $p \in (0, \infty)$, $q \in (0, \infty]$, $r \in \mathbf{R}$, $n \in \mathbf{N}$, and $x \in s$, then $H_q^p(rx) = rH_q^p(x)$ and $\xi_n \circ H_q^p = H_q^p \circ \xi_n$.*

PROPOSITION 2.2: *$H_q^p$ is continuous for every $p \in (0, \infty)$ and $q \in (0, \infty]$.*

*Proof:* The expression $\sqrt[p]{|\xi_n(x)|_q^p - |\xi_{n-1}(x)|_q^p}$ is a continuous function on $s$ since it depends on only finitely many $x_i$'s. In addition, if $x_n = 0$ then $|\xi_n(x)|_q = |\xi_{n-1}(x)|_q$ so the expression vanishes.

Since $|\xi_n(H_q^p(x))|_p = |\xi_n(x)|_q$ for every $x \in s$ and $n \in \mathbf{N}$ we have

PROPOSITION 2.3: *$|H_q^p(x)|_p = |x|_q$ for every $x \in s$, $p \in (0, \infty)$, and $q \in (0, \infty]$.*

PROPOSITION 2.4: *$H_p^p$ is the identity and $H_q^p \circ H_r^q = H_r^p$ for every $p, q \in (0, \infty)$ and $r \in (0, \infty]$.*

*Proof:* The identity shares with $H_p^p$ the properties of preserving sign and preserving the $p$-norms of projections onto the first $n$-coordinates. The second statement follows by a similar argument: the composition of sign preserving maps is sign preserving and for each $x \in s$ and $n \in \mathbf{N}$, $|\xi_n(H_q^p \circ H_r^q(x))|_p = |\xi_n(H_r^q(x))|_q = |\xi_n(x)|_r$.

Combining the last three propositions we find:

THEOREM 2.5: *If $p, q \in (0, \infty)$, then $H_q^p\colon s \to s$ is a homeomorphism that preserves norm, i.e., $|H_q^p(x)|_p = |x|_q$ for every $x \in s$.*

Mazur [5] constructed homeomorphisms $G_q^p\colon s \to s$ such that $G_q^p(\ell^q) = \ell^p$. However, Mazur's homeomorphisms are not normpreserving.

## 3. The sup norm

We now turn to the main topic of this paper: finding homeomorphisms $h\colon s \to s$ with the property $|h(x)|_p = |x|_\infty$. In other words, proving that the sup norm on $s$ is topologically indistiguishable from, for instance, the Hilbert norm.

In view of Theorem 2.5 we may restrict ourselves to the case $p = 1$. Note that if every $x_n$ is an element of $[-1, 1]$, then $H_\infty^1(1, x_2, x_3, \ldots) = (1, 0, 0, \ldots)$. So $H_\infty^1$

is norm preserving but not a homeomorphism. We shall prove, however, that $H^1_\infty$ can be approximated by norm preserving homeomorphisms. In the remainder of this paper we will denote $H^1_\infty$ simply by $H$. Let us recall that $H$ has the following properties: if $y = H(x)$ and $n \in \mathbf{N}$, then

$$\begin{aligned} |y|_1 &= \sum_{i=1}^{\infty} |y_i| = \sup\{|x_1|, |x_2|, \ldots\} = |x|_\infty, \\ |\xi_n(y)|_1 &= \sum_{i=1}^{n} |y_i| = \max\{|x_1|, \ldots, |x_n|\} = |\xi_n(x)|_\infty, \\ y_n &= \operatorname{sgn}(x_n)(|\xi_n(x)|_\infty - |\xi_{n-1}(x)|_\infty) \\ &= \operatorname{sgn}(x_n)\max\{0, |x_n| - |\xi_{n-1}(x)|_\infty\}. \end{aligned}$$

Let us explore the fibres of $H$:

PROPOSITION 3.1: *If $y \in s$, then*

$$H^{-1}(y) = \prod_{y_n \neq 0} \{\operatorname{sgn}(y_n)|\xi_n(y)|_1\} \times \prod_{y_n = 0} [-|\xi_n(y)|_1, |\xi_n(y)|_1].$$

*Proof:* Let $H(x) = y$ and $n \in \mathbf{N}$. First, we have $|x_n| \leq |\xi_n(x)|_\infty = |\xi_n(y)|_1$ so $x_n \in [-|\xi_n(y)|_1, |\xi_n(y)|_1]$. If $y_n \neq 0$, then by the formula above $|x_n| > |\xi_{n-1}(x)|_\infty$ and hence $|x_n| = |\xi_n(x)|_\infty = |\xi_n(y)|_1$. Since $H$ is sign preserving we have $x_n = \operatorname{sgn}(y_n)|\xi_n(y)|_1$.

Now let $x$ be such that $x_n = \operatorname{sgn}(y_n)|\xi_n(y)|_1$ if $y_n \neq 0$ and $|x_n| \leq |\xi_n(y)|_1$ if $y_n = 0$. We shall prove by induction that $|\xi_n(x)|_\infty = |\xi_n(y)|_1$ for every $n$. Since we also have that $y_n \neq 0$ implies $\operatorname{sgn}(y_n) = \operatorname{sgn}(x_n)$, it follows that $H(x) = y$. We have $|\xi_0(x)|_\infty = |\xi_0(y)|_1 = 0$. Let $|\xi_{n-1}(x)|_\infty = |\xi_{n-1}(y)|_1$ and consider

$$|\xi_n(x)|_\infty = \max\{|x_n|, |\xi_{n-1}(x)|_\infty\} = \max\{|x_n|, |\xi_{n-1}(y)|_1\}.$$

If $y_n = 0$ then $|\xi_{n-1}(y)|_1 = |\xi_n(y)|_1 \geq |x_n|$ and if $y_n \neq 0$ then $|x_n| = |\xi_n(y)|_1 \geq |\xi_{n-1}(y)|_1$, so we may conclude that $|\xi_n(x)|_\infty = |\xi_n(y)|_1$.

A continuous map $f\colon X \to Y$ is called **proper** if the preimage of every compact subset of $Y$ is a compact subset of $X$.

LEMMA 3.2: *$H\colon s \to s$ is a proper surjection.*

*Proof:* According to 3.1 every fibre is nonempty.

To show that $H$ is proper consider a compactum $C$ in $s$. Select a sequence of positive real numbers $M = (M_1, M_2, \ldots)$ such that

$$C \subset \prod_{n=1}^{\infty} [-M_n, M_n].$$

If $x$ and $y$ are such that $H(x) = y \in C$, then for each $n$ we have $|x_n| \le |\xi_n(x)|_\infty = |\xi_n(y)|_1 \le |\xi_n(M)|_1$. Consequently,

$$H^{-1}(C) \subset \prod_{n=1}^{\infty} [-|\xi_n(M)|_1, |\xi_n(M)|_1]$$

and hence $H^{-1}(C)$ is compact by continuity of $H$.

## 4. Shrinking

We need some definitions. A proper surjection $f: X \to Y$ is called a **near homeomorphism** if for every open covering $\mathcal{V}$ of $Y$ there is a homeomorphism $h: X \to Y$ such that $f$ and $h$ are $\mathcal{V}$-close, i.e. for every $x \in X$ there is a $V \in \mathcal{V}$ with $\{f(x), h(x)\} \subset V$. A proper surjection $f: X \to Y$ is called **shrinkable** if for every pair of open coverings $\mathcal{U}$ and $\mathcal{V}$ of $X$ respectively $Y$ there is a (shrinking) homeomorphism $h: X \to X$ such that $f \circ h$ and $f$ are $\mathcal{V}$-close and the collection of fibres of $f \circ h$ refines $\mathcal{U}$. The key to showing that we can approximate $H$ with norm preserving homeomorphisms is the Bing Shrinking Criterion (see [3] or [7]):

THEOREM 4.1: *A map between complete spaces is a near homeomorphism if and only if it is shrinkable.*

For $r \in [0, \infty)$ we define the compacta $A_r = \{x \in s : |x|_\infty \le r\}$ and $B_r = \{x \in s : |x|_1 \le r\}$. Compactness follows because $A_r = [-r, r]^{\mathbf{N}}$ and $B_r$ is a closed subset of $A_r$ by lower semi-continuity of $p$-norms. We intend to show that $H$ is shrinkable by homeomorphisms $h: s \to s$ that leave the sup norm invariant, i.e., $|h(x)|_\infty = |x|_\infty$ for every $x \in s$. Consequently, we have $h(A_r) = A_r$ for every $r \ge 0$. Since $|H(x)|_1 = |x|_\infty$ for all $x \in s$ and $H$ is a surjection, we also have $H(A_r) = B_r$ for every $r \ge 0$. In the standard proof for Bing's Theorem the homeomorphism $g$ that approximates $H$ is obtained as a limit of a sequence of maps of the form $g_n = H \circ h_n \circ \ldots \circ h_1$, $n \in \mathbf{N}$, where every $h_i$ is a shrinking homeomorphism. Since we may assume that every $g_n$ has the property $g_n(A_r) = B_r$, we may conclude by the compactness of $A_r$ that $g(A_r) = B_r$ for each $r \ge 0$. This result implies that $|g(x)|_1 = |x|_\infty$ for each $x \in s$ because $g$ is a homeomorphism. In addition, note that if the shrinking homeomorphisms are sign preserving, then also $g$ is sign preserving.

LEMMA 4.2: *If $f: X \to Y$ is a proper surjection between metric spaces, $d$ is a metric on $Y$ and $\delta: X \to (0, \infty)$ and $\eta: Y \to (0, \infty)$ are lower semi-continuous functions, then there exists a (Lipschitz) map $\varepsilon: Y \to (0, \infty)$ such that $\varepsilon(f(x)) \le \delta(x)$ for every $x \in X$, $\varepsilon(y) \le \eta(y)$ and $|\varepsilon(y) - \varepsilon(y')| \le d(y, y')$ for all $y, y' \in Y$.*

*Proof:* For each $y \in Y$ we may define

$$\zeta(y) = \min\{\delta(x) : f(x) = y\} \in (0, \infty)$$

because fibres are compact and nonempty and $\delta$ is lower semi-continuous. We show that $\zeta$ is lower semi-continuous. Let $t > 0$ and assume that $\zeta(y) > t$. Define the closed set $F = \delta^{-1}((0,t])$ and note that $F$ is disjoint from the fibre of $y$. Every proper map between metric spaces is closed, so $O = Y \setminus f(F)$ is a neighbourhood of $y$. Note that $O \subset \zeta^{-1}((t,\infty))$, proving lower semi-continuity of $\zeta$. Define $\zeta'\colon Y \to (0,\infty)$ by $\zeta'(y) = \min\{\eta(y), \zeta(y)\}$ and note that $\zeta'$ is lower semi-continuous as well.

We now define $\varepsilon\colon Y \to \mathbf{R}$:

$$\varepsilon(y) = \sup\{r \in \mathbf{R} : \zeta'(z) \geq r \text{ for all } z \in Y \text{ with } d(z,y) < r\}.$$

Obviously, $\varepsilon(y) \leq \zeta'(y) \leq \min\{\eta(y), \delta(x)\}$ for every $x$ and $y$ with $f(x) = y$. We now verify that $\varepsilon$ is a positive function. If $y \in Y$, then by lower semi-continuity there is an $r > 0$ such that $\zeta'(z) \geq \zeta'(y)/2$ for all $z \in Y$ with $d(z,y) < r$. Note that $\varepsilon(y) \geq \min\{r, \zeta'(y)/2\} > 0$.

Let $y, y' \in Y$ and put $r = \varepsilon(y) - d(y,y')$. Observe that if $d(z,y') < r$, then $d(z,y) < \varepsilon(y)$ and hence $\zeta'(z) \geq \varepsilon(y) > r$. So $\varepsilon(y') \geq r$, which means $\varepsilon(y) - \varepsilon(y') \leq d(y,y')$, proving that $\varepsilon$ has Lipschitz constant no greater than 1.

On $s$ we will use the following F-norm: for $x \in s$,

$$\|x\| = \max\{\min\{1/n, |x_n|\} : n \in \mathbf{N}\}.$$

Note that this norm is bounded by 1 and that the corresponding metric $d(x,y) = \|x - y\|$ generates the product topology on $s = \mathbf{R}^{\mathbf{N}}$. In addition, we have $\|x\| \leq |x|_\infty \leq |x|_1$ for each $x \in s$.

For the sake of clarity we will denote the domain of $H$ by $X$ and the range of $H$ by $Y$. The basic shrinking of fibres of $H$ is done in the following lemma, the proof of which occupies the remainder of this section.

Lemma 4.3: *Let $F$ and $G$ be disjoint closed subsets of $Y$, let $\varepsilon\colon Y \to (0,1]$ be a continuous map, and let $n \in \mathbf{N}$. Then there exists a sign preserving homeomorphism $g\colon X \to X$ with the following properties: if $x$ and $x'$ are arbitrary points of $X$ such that $H(x) = H(x')$ and if $i \in \mathbf{N}$, then*

(1) $|H \circ g(x) - H(x)|_1 \leq \varepsilon(H(x))$,

(2) *$H(x) \in G$ implies $g(x) = x$,*

(3) *$i \geq n+1$ implies $|\xi_i \circ g(x)|_\infty = |\xi_i(x)|_\infty$,*

(4) $i \geq n+2$ *implies* $\pi_i \circ H \circ g = \pi_i \circ H$,
(5) $|g(x)|_\infty = |x|_\infty$,
(6) $i \geq n+2$ *implies* $\pi_i \circ g = \pi_i$,
(7) $H \circ g(x) = H \circ g(x')$,
(8) $|\pi_i(g(x) - g(x'))| \leq (1 + \varepsilon(H(x)))|\pi_i(x - x')|$,
(9) $H(x) \in F$ *implies* $|\pi_{n+1}(g(x) - g(x'))| \leq \varepsilon(H(x))$.

The shrinking properties of $g$ are expressed by (8) and (9): $g$ shrinks the $(n+1)$-projection of all $H$-fibres of points in $F$, while at the same time not significantly expanding the projection of fibres in other coordinates.

Let $I = [0,1]$ be the unit interval and let $\Gamma$ be the following subspace of $\mathbf{R}^2$:

$$\Gamma = (\{1\} \times I) \cup (I \times \{1\}) = \{(u,v) : u, v \geq 0 \text{ and } \max\{u,v\} = 1\}.$$

Put $L = \{(0,1)\} \cup ((0,1/2] \times (0,1]) \subset I^2$. We shall construct an isotopy $\alpha_{t,r}$, $(t,r) \in L$, of the arc $\Gamma$. $\alpha_{t,r}$ is a PL-map that will keep the segment $[0,1-t] \times \{1\}$ fixed, that shrinks the segment $\{1\} \times I$ linearly to $\{1\} \times [0,r]$, and that expands the segment $[1-t,1] \times \{1\}$ uniformly into the arc $([1-t,1] \times \{1\}) \cup (\{1\} \times [r,1])$. Formally, if we denote the two components of $\alpha_{t,r}$ by $(p_{t,r}, q_{t,r})$, then for $(t,r) \in L$ and $(u,v) \in \Gamma$ we have

$$p_{t,r}(u,1) = u \quad \text{and} \quad q_{t,r}(u,1) = 1 \quad \text{for } u \leq 1-t,$$
$$p_{t,r}(1,v) = 1 \quad \text{and} \quad q_{t,r}(1,v) = rv,$$

and for $t > 0$ and $u \geq 1-t$,

$$p_{t,r}(u,1) = \min\left\{1, 1 - t + \frac{t+1-r}{t}(u - 1 + t)\right\},$$
$$q_{t,r}(u,1) = \min\left\{1, 1 + t - \frac{t+1-r}{t}(u - 1 + t)\right\}.$$

The continuity of $\alpha_{t,r}$ is obvious except perhaps at the point $(t,r) = (0,1)$, where $\alpha_{t,r}$ is the identity. Note that if $0 < t < \varepsilon$ and $1 - \varepsilon < r < 1$, then we have no problem if $u \leq 1-t$ or $u = 1$. Consider the remaining case $1 - t < u < 1$. Then $1-t \leq p(u,1) \leq 1$ so $p_{t,r}$ is continuous. Note that we have that $0 < (u-1+t)/t < 1$ and hence $0 < (t+1-r)(u-1+t)/t < 2\varepsilon$. Consequently, $1 - 2\varepsilon \leq q_{t,r} \leq 1$ so $q_{t,r}$ is continuous as well. It obvious that the inverse $\alpha_{t,r}^{-1} = (\hat{p}_{t,r}, \hat{q}_{t,r})$ depends continuously on $t$ and $r$ as well.

LEMMA 4.4: *If $(t,r) \in L$ and $(u,v) \in \Gamma$, then $u \leq p_{t,r}(u,v) \leq u/(1-t)$.*

*Proof:* If $t = 0$ or if $u \leq 1 - t < 1$, then $u = p_{t,r}(u,v) \leq u/(1-t)$.

Let $u > 1-t$. Then $p_{t,r}(u,v) \le 1 \le u/(1-t)$. If $v < 1$, then $u = 1$ and $p_{t,r}(u,v) = 1 \ge u$. On the other hand, if $v = 1$, then since $(t+1-r)/t \ge 1$ we have

$$p_{t,r}(u,v) = \min\{1, 1-t+\frac{t+1-r}{t}(u-1+t)\} \ge \min\{1, 1-t+u-1+t\} = u.$$

We will now start the process of defining $g\colon X \to X$. Let $F$, $G$, $\varepsilon$, and $n$ be as in the premise of 4.3. In view of 4.2 we may assume that $|\varepsilon(y)-\varepsilon(y')| \le \|y-y'\|$ for $y, y' \in Y$.

Let $x = (x_1, x_2, \ldots) \in X$ and put $y = (y_1, y_2, \ldots) = H(x)$. We also put

$$a = |\xi_{n+1}(y)|_1 \quad \text{and} \quad b = |\xi_n(y)|_1.$$

Using Proposition 2.3 and the definition of $H$ we find:

CLAIM 4.5: *$a$ and $b$ depend continuously on $y$ and on $x$ and*

(1) $a = |\xi_{n+1}(x)|_\infty$ *and* $b = |\xi_n(x)|_\infty$,
(2) $a = \max\{b, |x_{n+1}|\} \ge b$.
(3) $|y_{n+1}| = a - b$.

Assume that $b \ne 0$. We define $\tilde{y} \in Y$ by

$$\tilde{y}_i = \begin{cases} ay_i/b, & \text{for } i \le n, \\ 0, & \text{for } i = n+1, \\ y_i, & \text{for } i \ge n+2. \end{cases}$$

So $\tilde{\ }$ is a continuous mapping from $Y \setminus \xi_n^{-1}(\mathbf{0})$ to itself. Note that $|\xi_{n+1}(\tilde{y})|_1 = |\xi_n(\tilde{y})|_1 = |\frac{a}{b}\xi_n(y)|_1 = a$, so $a$ depends continuously on $\tilde{y}$. We then obviously have

$$|\xi_i(y)|_1 = |\xi_i(\tilde{y})|_1 \quad \text{for every } i \ge n+1.$$

Let the parameter $\lambda$ be the distance from $\tilde{y}$ to the set $G$:

$$\lambda = d(\tilde{y}, G),$$

and let $\mu$ be defined by

$$\mu = \min\left\{1, \frac{2\lambda}{\lambda + d(\tilde{y}, F)}\right\}.$$

Let the parameters $t$ and $r$ be given by

$$t = \frac{\lambda\varepsilon(\tilde{y})}{5\max\{1,a\}} \quad \text{and} \quad r = \min\left\{1, \frac{\varepsilon(\tilde{y})}{4\mu\max\{1,a\}}\right\}.$$

Finally, we will use the abbreviations

$$p = p_{t,r}(b/a, |x_{n+1}|/a) \quad \text{and} \quad q = q_{t,r}(b/a, |x_{n+1}|/a).$$

Since $b > 0$ and $a = \max\{b, |x_{n+1}|\}$, we have $(b/a, |x_{n+1}|/a) \in \Gamma$. Also, since $\lambda \leq 1$ and $\varepsilon(\tilde{y}) \leq 1$, $t \leq 1/5$ and if $t = 0$ then $\lambda = \mu = 0$ and $r = 1$. So $(t, r) \in L$ and the parameters $p$ and $q$ are well defined. Note that if $b \neq 0$, then $\lambda$, $\mu$, $t$, and $r$ depend continuously on $\tilde{y}$ and hence on $y$ and on $x$. We define $g(x)$ as follows: for $i \in \mathbf{N}$,

$$\pi_i \circ g(x) = \begin{cases} 0, & \text{for } i \leq n \text{ and } b = 0, \\ apx_i/b, & \text{for } i \leq n \text{ and } b \neq 0, \\ x_{n+1}, & \text{for } i = n+1 \text{ and } b = 0, \\ \operatorname{sgn}(x_{n+1})aq, & \text{for } i = n+1 \text{ and } b \neq 0, \\ x_i, & \text{for } i \geq n+2. \end{cases}$$

It is obvious that $g$ is sign preserving and that $g$ satisfies property (6).

Throughout the remainder of this section let $x' \in X$ be an arbitrary point with the property $H(x') = H(x) = y$. Since the parameters $a$, $b$, $\lambda$, $\mu$, $t$, and $r$ depend on $y$, their values for $x'$ are identical to those for $x$ (provided that these parameters are defined). Put, for $b \neq 0$, $p' = p_{t,r}(b/a, |x'_{n+1}|/a)$ and $q' = q_{t,r}(b/a, |x'_{n+1}|/a)$. Let $z = H \circ g(x)$ and $z' = H \circ g(x')$.

We shall give a number of useful conditional properties of $g$. Note that $x$ and $x'$ may be interchanged.

Claim 4.6: *If $b > 0$, then $\xi_n(z) = \frac{ap}{b}\xi_n(y)$ and*

$$z_{n+1} = \operatorname{sgn}(x_{n+1}) \max\{0, aq - ap\}.$$

*Proof:* Consider

$$\begin{aligned} \xi_n(z) &= \xi_n \circ H \circ g(x) = H \circ \xi_n \circ g(x) = H\Big(\frac{ap}{b}\xi_n(x)\Big) \\ &= \frac{ap}{b} H \circ \xi_n(x) = \frac{ap}{b}\xi_n \circ H(x) = \frac{ap}{b}\xi_n(y), \end{aligned}$$

where we used Proposition 2.1. Observe that

$$\begin{aligned} z_{n+1} &= \operatorname{sgn}(x_{n+1}) \max\{0, |\pi_{n+1} \circ g(x)| - |\xi_n \circ g(x)|_\infty\} \\ &= \operatorname{sgn}(x_{n+1}) \max\{0, aq - ap\}. \end{aligned}$$

Claim 4.7: *If $b < a$, then $a = |x_{n+1}|$ and $x_{n+1} = x'_{n+1}$ and if, in addition, $b > 0$, then $p = p'$ and $q = q'$.*

*Proof:* We use Claim 4.5: $a = \max\{b, |x_{n+1}|\} = |x_{n+1}|$. We also have $|y_{n+1}| = a - b > 0$ and hence $x_{n+1} = x'_{n+1}$ with Proposition 3.1. Consequently, $p = p'$ and $q = q'$ if $b > 0$.

CLAIM 4.8: *If* $b \leq a(1-t)$ *then* $g(x) = x$.

*Proof:* We always have $\pi_i \circ g(x) = x_i$ for all $i \geq n+2$.

Case I: $b = 0$. Then $\xi_n(x) = \mathbf{0} = \xi_n \circ g(x)$ and $\pi_{n+1} \circ g(x) = x_{n+1}$.

Case II: $0 < b \leq a(1-t)$ and $b < a$. So $b/a \leq 1-t$ and $|x_{n+1}| = a$ by Claim 4.7. Hence $p = b/a$ and $q = 1$ by the definition of $\alpha_{t,r}$. Note that substitution into the definition of $g$ produces $g(x) = x$.

Case III: $0 < b \leq a(1-t)$ and $b = a$. This forces $t = 0$ and hence $r = 1$ by the definition of $L$, so $p = 1$ and $q = |x_{n+1}|/a$. Substitution into the definition gives $g(x) = x$.

CLAIM 4.9: *If* $b \geq a(1-t)$ *and* $b > 0$, *then* $|\tilde{y} - y|_1 = 2(a-b) \leq \lambda\varepsilon(\tilde{y})/2$ *and* $\varepsilon(\tilde{y}) \leq 2\varepsilon(y)$.

*Proof:* Consider with Claim 4.5(c)

$$|y_{n+1}| = a - b \leq ta = \frac{\lambda a \varepsilon(\tilde{y})}{5\max\{1,a\}} \leq \lambda\varepsilon(\tilde{y})/5$$

and

$$|\tilde{y} - y|_1 = \sum_{i=1}^{n} \left|\frac{a}{b}y_i - y_i\right| + |y_{n+1}| = \frac{a-b}{b}b + a - b = 2(a-b) \leq \lambda\varepsilon(\tilde{y})/2.$$

With the Lipschitz property of $\varepsilon$ we have

$$|\varepsilon(\tilde{y}) - \varepsilon(y)| \leq \|\tilde{y} - y\| \leq |\tilde{y} - y|_1 \leq \varepsilon(\tilde{y})/2.$$

This means that $\varepsilon(\tilde{y}) \leq 2\varepsilon(y)$ and the claim is proved.

CLAIM 4.10 (Property (3)): *If* $i \geq n+1$ *then* $|\xi_i \circ g(x)|_\infty = |\xi_i(x)|_\infty$.

Note that the fact that $g$ preserves the sup norm (Property (5)) follows immediately from this claim.

*Proof:* We first consider the case $i = n+1$. If $b = 0$ then $|\xi_{n+1} \circ g(x)|_\infty = |x_{n+1}| = a = |\xi_{n+1}(x)|_\infty$. Let $b \neq 0$. Note that $|\xi_n \circ g(x)|_\infty = |\frac{ap}{b}\xi_n(x)|_\infty = ap$ and that $|\pi_{n+1} \circ g(x)| = aq$. So we have

$$|\xi_{n+1} \circ g(x)|_\infty = \max\{ap, aq\} = a = |\xi_{n+1}(x)|_\infty$$

because $(p,q) \in \Gamma$.

Now let $i \geq n+2$. We have

$$\begin{aligned} |\xi_i \circ g(x)|_\infty &= \max\{|\xi_{n+1} \circ g(x)|_\infty, |\pi_{n+2} \circ g(x)|, \ldots, |\pi_i \circ g(x)|\} \\ &= \max\{|\xi_{n+1}(x)|_\infty, |x_{n+2}|, \ldots, |x_i|\} = |\xi_i(x)|_\infty. \end{aligned}$$

CLAIM 4.11 (Property (4)): *If* $i \geq n+2$ *then* $z_i = y_i$.

*Proof:* This statement follows directly from Claim 4.8:

$$\begin{aligned} z_i = \pi_i \circ H \circ g(x) &= \operatorname{sgn}(\pi_i \circ g(x))(|\xi_i \circ g(x)|_\infty - |\xi_{i-1} \circ g(x)|_\infty) \\ &= \operatorname{sgn}(x_i)(|\xi_i(x)|_\infty - |\xi_{i-1}(x)|_\infty) = y_i. \end{aligned}$$

CLAIM 4.12 (Property (1)): $|z-y|_1 \leq \varepsilon(y)$.

*Proof:* According to Claim 4.8, $b \leq a(1-t)$ forces $g(x) = x$, so we may assume that $b > a(1-t)$. In view of Claim 4.11 we have

$$|z-y|_1 = |H \circ g(x) - H(x)|_1 = |\xi_{n+1}(z-y)|_1 = |\xi_n(z-y)|_1 + |z_{n+1} - y_{n+1}|.$$

We have with Lemma 4.4 that $b \leq ap \leq b/(1-t)$ and hence, using Claim 4.6,

$$\begin{aligned} |\xi_n(z-y)|_1 &= \left(\frac{ap}{b} - 1\right)|\xi_n(y)|_1 = \left(\frac{ap}{b} - 1\right) b \\ &= ap - b \leq \frac{b}{1-t} - b = \frac{tb}{1-t}. \end{aligned}$$

Since $t \leq \varepsilon(\tilde{y})/(5a)$, $b \leq a$, and $t \leq \varepsilon(\tilde{y})/5$ we have

$$|\xi_n(z-y)|_1 \leq \frac{\varepsilon(\tilde{y})b}{5a(1-t)} \leq \frac{\varepsilon(\tilde{y})}{5(1-t)} \leq \frac{\varepsilon(\tilde{y})}{5-\varepsilon(\tilde{y}))} \leq \varepsilon(\tilde{y})/4.$$

Now we consider $z_{n+1} = \operatorname{sgn}(x_{n+1}) \max\{0, aq - ap\}$ (Claim 4.6).

Case I: $p \geq q$. Since $(p,q) \in \Gamma$ we have $p = 1$, which leads to $b \geq a - at$ in view of Lemma 4.4. So we have $|y_{n+1}| = a - b \leq at \leq \varepsilon(\tilde{y})/5$. Note that since $aq - ap \leq 0$ we have $z_{n+1} = 0$. So

$$|z_{n+1} - y_{n+1}| = |y_{n+1}| \leq \varepsilon(\tilde{y})/5.$$

Case II: $q > p$. So we have $q = 1$ and $z_{n+1} = \operatorname{sgn}(x_{n+1})(a - ap)$. Since $y_{n+1} = \operatorname{sgn}(x_{n+1})(a-b)$ we may conclude that

$$|z_{n+1} - y_{n+1}| = |a - ap - a + b| = |b - ap| = |\xi_n(z-y)|_1 \leq \varepsilon(\tilde{y})/4.$$

So in both cases we arrive at $|z-y|_1 \leq \varepsilon(\tilde{y})/2 \leq \varepsilon(y)$, where we used Claim 4.9.

CLAIM 4.13: *If* $b > 0$ *then* $\tilde{z} = \tilde{y}$.

*Proof:* We have $\pi_{n+1}(\tilde{z}) = 0 = \pi_{n+1}(\tilde{y})$ and if $i \geq n+2$ then $\pi_i(\tilde{z}) = z_i = y_i = \pi_i(\tilde{y})$ by Claim 4.11.

Since we are in the case $b \neq 0$, we find with Claim 4.6 that $\xi_n(z) = \frac{ap}{b}\xi_n(y)$. Note that by Lemma 4.4, $p \geq b/a > 0$, so we have $|\xi_n(z)|_1 = ap \geq bp > 0$. With Claim 4.8 and Proposition 2.3 we find

$$\begin{aligned}\xi_n(\tilde{z}) &= \frac{|\xi_{n+1}(z)|_1}{|\xi_n(z)|_1}\xi_n(z) = \frac{|\xi_{n+1}\circ g(x)|_\infty}{ap}\frac{ap}{b}\xi_n(y) \\ &= \frac{|\xi_{n+1}(x)|_\infty}{b}\xi_n(y) = \frac{|\xi_{n+1}(y)|_1}{b}\xi_n(y) = \frac{a}{b}\xi_n(y) = \xi_n(\tilde{y}).\end{aligned}$$

CLAIM 4.14: *$g$ is a homeomorphism.*

*Proof:* We first show that $g(x)$ depends continuously on $x$. Since $x_{n+1} = 0$ implies $q = 0$, the continuity of $g$ is obvious at every point outside of $\xi_n^{-1}(\mathbf{0})$. Let $w \in X$ be a fixed point such that $\xi_n(w) = \mathbf{0}$ and hence $\xi_n \circ g(w) = \mathbf{0}$ and $\pi_{n+1}\circ g(w) = w_{n+1}$. Since $g$ is essentially a map from $\mathbf{R}^{n+1}$ to itself, we may use the sup norm to establish continuity. Let $\delta = |\xi_{n+1}(x-w)|_\infty$. Note that $b \leq \delta$ and that $|x_{n+1} - w_{n+1}| \leq \delta$.

If $w_{n+1} = 0$ then, by Claim 4.10, $|\xi_{n+1}(g(x) - g(w))|_\infty = |\xi_{n+1}\circ g(x)|_\infty = |\xi_{n+1}(x)|_\infty = \delta$. If $b = 0$ then $|\xi_{n+1}(g(x) - g(w))|_\infty = |x_{n+1} - w_{n+1}| \leq \delta$.

Let $b > 0$ and $w_{n+1} \neq 0$. According to Lemma 4.4 we have

$$p \leq b/(a(1-t)) \leq 2b/a.$$

Without loss of generality we may assume that $3\delta < |w_{n+1}|$ and hence that $a \geq |x_{n+1}| \geq |w_{n+1}| - \delta > 2\delta \geq 2b$. So we have $p < 1$ and, since $1 = \max\{p, q\}$, $q$ must equal 1. In addition, we found that $a > b$ which implies $a = |x_{n+1}|$. So we have

$$|\pi_{n+1}(g(x) - g(w))| = |\operatorname{sgn}(x_{n+1})a - w_{n+1}| = |x_{n+1} - w_{n+1}| \leq \delta.$$

For the remaining case $i \leq n$ we obtain

$$|\xi_n(g(x) - g(w))|_\infty = |\xi_n \circ g(x)|_\infty = ap \leq 2b \leq 2\delta.$$

To show that $g$ is a homeomorphism we will display its inverse. Let $\hat{g}\colon X \to X$ be defined in the same manner as $g$ with the function $\alpha_{t,r} = (p_{t,r}, q_{t,r})$ replaced with its inverse $\alpha_{t,r}^{-1} = (\hat{p}_{t,r}, \hat{q}_{t,r})$. Put $\hat{x} = g(x)$ and $z = H(\hat{x})$. The function $\hat{g}$ is obviously continuous as well and to prove that $\hat{g}(\hat{x}) = x$ we only have to consider the first $n+1$ coordinates.

Case I: $b = 0$. Then $\xi_n(\hat{x}) = \xi_n(x) = \mathbf{0}$ and $\hat{x}_{n+1} = x_{n+1}$, so $\hat{g}(\hat{x}) = x$ is obviously true.

Case II: $b > 0$. Then, by Claim 4.13, $\tilde{z} = \tilde{y}$ so $\hat{x}$ has the same values for the parameters $a$, $\lambda$, $\mu$, $t$, and $r$ as $x$. Let $\hat{b} = |\xi_n(z)|_1 = |\xi_n(\hat{x})|_\infty$ and note that, as in Claim 4.13, we have $\hat{b} = ap > 0$. Observe that $|\hat{x}_{n+1}| = aq$. Consider $\hat{p} = \hat{p}_{t,r}(\hat{b}/a, |\hat{x}_{n+1}|/a) = \hat{p}_{t,r}(p,q)$ and $\hat{q} = \hat{q}_{t,r}(\hat{b}/a, |\hat{x}_{n+1}|/a) = \hat{q}_{t,r}(p,q)$. Since $(\hat{p}_{t,r}, \hat{q}_{t,r})$ is the inverse of $(p_{t,r}, q_{t,r})$ we have $\hat{p} = b/a$ and $\hat{q} = |x_{n+1}|/a$. So

$$\xi_n \circ \hat{g}(\hat{x}) = \frac{a\hat{p}}{\hat{b}}\xi_n(\hat{x}) = \frac{ab/a}{ap}\frac{ap}{b}\xi_n(x) = \xi_n(x)$$

and

$$\pi_{n+1} \circ \hat{g}(\hat{x}) = \operatorname{sgn}(\hat{x}_{n+1})a\hat{q} = \operatorname{sgn}(x_{n+1})|x_{n+1}| = x_{n+1}.$$

So $\hat{g} \circ g$ is the identity function and, by symmetry, $g \circ \hat{g}$ is the identity as well.

CLAIM 4.15 (Property (2)): *If $y \in G$ then $g(x) = x$.*

*Proof:* In view of Claim 4.8 we may assume that $b > a(1-t)$. According to Claim 4.9, $\|\tilde{y} - y\| \leq |\tilde{y} - y|_1 \leq \lambda/2 = \frac{1}{2}d(\tilde{y}, G)$.

Case I: $\lambda = 0$. Note that $\mu = 0$ and $r = 1$, which means that $\alpha_{t,r}$ is the identity and $g(x) = x$.

Case II: $\lambda > 0$. Then $d(\tilde{y}, y) < \lambda = d(\tilde{y}, G)$ and we have $y \notin G$.

CLAIM 4.16 (Property (9)): *If $y \in F$ then $|\pi_{n+1}(g(x) - g(x'))| \leq \varepsilon(y)$.*

*Proof:* Case I: $b = 0$. Then $\xi_n(x) = \xi_n(x') = \mathbf{0}$ so $y_{n+1} = x_{n+1} = x'_{n+1}$. Consequently, $\pi_{n+1} \circ g(x) = y_{n+1} = \pi_{n+1} \circ g(x')$.

Case II: $0 < b < a$. By Claim 4.7 we have $x_{n+1} = x'_{n+1}$ and hence $q = q'$. So

$$\pi_{n+1} \circ g(x) = \operatorname{sgn}(x_{n+1})aq = \operatorname{sgn}(x'_{n+1})aq' = \pi_{n+1} \circ g(x').$$

Case III: $b = a > 0$ and $\lambda = 0$. Then we have $\tilde{y} = y$ in view of Claim 4.9. On the other hand, $\lambda = 0$ means $\tilde{y} \in G$ so $\tilde{y} = y \notin F$.

Case IV: $b = a > 0$ and $\lambda > 0$. According to Claim 4.9 we have $d(\tilde{y}, y) \leq \|\tilde{y} - y\| \leq \lambda/2$. Since $y \in F$ we have $d(\tilde{y}, F) \leq \lambda/2$ and hence $\mu = 1$ and $r \leq \varepsilon(\tilde{y})/(4a)$. Since $b/a = 1$ we have $q = r|x_{n+1}|/a$ and $q' = r|x'_{n+1}|/a$. So

$$\begin{aligned}|\pi_{n+1}(g(x) - g(x'))| &= |rx_{n+1} - rx'_{n+1}| \leq \frac{\varepsilon(\tilde{y})}{4a}|x_{n+1} - x'_{n+1}| \\ &\leq \frac{\varepsilon(\tilde{y})}{4a}2a \leq \varepsilon(y)\end{aligned}$$

because $r < \varepsilon(\tilde{y})/(4a)$ and $\varepsilon(\tilde{y}) \leq 2\varepsilon(y)$ by Claim 4.9.

CLAIM 4.17 (Property (7)): $z = H \circ g(x) = H \circ g(x') = z'$.

*Proof:* In view of Claim 4.11 we have $z_i = y_i = z'_i$ for any $i \geq n+2$.

Case I: $b = 0$. Then $\xi_n(z) = \mathbf{0} = \xi_n(z')$ and $z_{n+1} = y_{n+1} = z'_{n+1}$.

Case II: $0 < b < a$. Then, by Claim 4.7, $x_{n+1} = x'_{n+1}$, $p = p'$, and $q = q'$, which means in view of Claim 4.6 that

$$\xi_n(z) = \frac{ap}{b}\xi_n(y) = \frac{ap'}{b}\xi_n(y) = \xi_n(z')$$

and

$$\begin{aligned} z_{n+1} &= \operatorname{sgn}(x_{n+1})\max\{0, aq - ap\} = \operatorname{sgn}(x'_{n+1})\max\{0, aq' - ap'\} \\ &= z'_{n+1}. \end{aligned}$$

Case III: $0 < b = a$. Then $p = p_{t,r}(1, |x_{n+1}|/a) = 1 \geq q$ and, by symmetry, $p' = 1 \geq q'$. If we combine this observation with the same formulas from Claim 4.6, then we obtain $\xi_n(z) = \frac{a}{b}\xi_n(y) = \xi_n(z')$ and $z_{n+1} = 0 = z'_{n+1}$.

CLAIM 4.18 (Property (8)): $|\pi_i(g(x) - g(x'))| \leq (1 + \varepsilon(y))|\pi_i(x - x')|$ *for every* $i \in \mathbf{N}$.

*Proof:* In view of property (6) and Claim 4.8 we may assume that $i \leq n+1$ and $b > a(t-1)$, so we have $\varepsilon(\tilde{y}) \leq 2\varepsilon(y)$ by Claim 4.9.

Case I: $b < a$. With Claim 4.7 we have $x_{n+1} = x'_{n+1}$, $p = p'$ and $q = q'$, so if $i \leq n$ then

$$|\pi_i(g(x) - g(x'))| = \left|\frac{ap}{b}x_i - \frac{ap'}{b}x'_i\right| = \frac{ap}{b}|x_i - x'_i|.$$

According to Lemma 4.4, $ap \leq b/(1-t)$ so we have

$$\frac{ap}{b} \leq \frac{1}{1-t} \leq \frac{1}{1 - \frac{1}{5}\varepsilon(\tilde{y})} = 1 + \frac{\varepsilon(\tilde{y})}{5 - \varepsilon(\tilde{y})} \leq 1 + \frac{\varepsilon(\tilde{y})}{4} \leq 1 + \frac{\varepsilon(y)}{2}.$$

Consequently, we obtain

$$|\pi_i(g(x) - g(x'))| \leq \left(1 + \frac{\varepsilon(y)}{2}\right)|x_i - x'_i|.$$

For $i = n+1$ we have

$$\pi_{n+1} \circ g(x) = \operatorname{sgn}(x_{n+1})aq = \operatorname{sgn}(x'_{n+1})aq' = \pi_{n+1} \circ g(x').$$

Case II: $b = a$. Then we have $p = 1$ and $q = r|x_{n+1}|/a$. If $i \leq n$, this means that $|\pi_i(g(x) - g(x'))| = |x_i - x'_i|$. For $i = n+1$, we have $\pi_{n+1} \circ g(x) = \operatorname{sgn}(x_{n+1})ar|x_{n+1}|/a = rx_{n+1}$ and hence

$$|\pi_{n+1}(g(x) - g(x'))| = |rx_{n+1} - rx'_{n+1}| \leq |x_{n+1} - x'_{n+1}|.$$

## 5. Conclusion

We now present our main result.

THEOREM 5.1: *For every $p \in (0,\infty)$ and every open covering $\mathcal{V}$ of $s$ there exists a sign preserving homeomorphism $g\colon s \to s$, $\mathcal{V}$-close to $H^p_\infty$, such that $|g(x)|_p = |x|_\infty$ for every $x \in s$.*

Theorem 5.1 is obtained by combining Theorem 2.5 with the following result:

THEOREM 5.2: *For every open covering $\mathcal{V}$ of $s$ there exists a sign preserving homeomorphism $g\colon s \to s$, $\mathcal{V}$-close to $H$, such that $|g(x)|_1 = |x|_\infty$ for every $x \in s$.*

*Proof:* As in §4 we consider $H$ to be a function from $X$ to $Y$. Let $\delta\colon X \to (0,1]$ and $\eta\colon Y \to (0,1]$ be arbitrary continuous functions. According to Lemma 4.2, we may assume that $\eta \circ H(x) \le \delta(x)/4$ for each $x \in X$ and that $|\eta(y)-\eta(z)| \le \|y-z\|$ for all $y, z \in Y$. Define for each $n \in \mathbf{N}$ the following pair of disjoint closed subsets of $Y$:

$$F_n = \{y \in Y : \eta(y) \le 4/(3n)\} \quad \text{and} \quad G_n = \{y \in Y : \eta(y) \ge 2/n\}.$$

Select a $\rho > 0$ such that

$$\sum_{n=1}^{\infty} a_n \le \frac{1}{3}, \quad \text{where } a_n = \rho^n \prod_{i=1}^{n-1}(1+\rho^i).$$

For instance, $\rho = 1/5$ works. Put $\varepsilon_j = \rho^j \eta$ for $j \in \omega$. For every $n \in \mathbf{N}$ use Lemma 4.3 to find a sign preserving homomorphism $g_n\colon X \to X$ that satisfies properties (1)–(9) for $F_n$, $G_n$, $\varepsilon_n$, and $n$.

Let $f_0$ be the identity on $X$ and $f_n = g_n \circ f_{n-1}$ for $n \in \mathbf{N}$. Then we have that every $f_n$ is a sign preserving homeomorphism. Consider an $x \in X$ and an $n \in \mathbf{N}$ and put $y(j) = H \circ f_j(x)$ for $j \in \omega$. Then by property (1) of $g_n$, we have $|y(n) - y(n-1)|_1 \le \varepsilon_n(y(n-1))$. Consider

$$\begin{aligned}|\varepsilon_n(y(n)) - \varepsilon_n(y(n-1))| &\le \rho^n \|y(n) - y(n-1)\| \le \rho^n |y(n) - y(n-1)|_1 \\ &\le \rho^n \varepsilon_n(y(n-1)).\end{aligned}$$

So we have for each $n \in \mathbf{N}$,

$$\varepsilon_n(y(n)) \le (1+\rho^n)\varepsilon_n(y(n-1)) = \rho(1+\rho^n)\varepsilon_{n-1}(y(n-1)).$$

By iteration we arrive at

$$\begin{aligned}\varepsilon_{n-1}(y(n-1)) &\leq \rho(1+\rho^{n-1})\varepsilon_{n-2}(y(n-2))\\ &\leq \rho(1+\rho^{n-1})\rho(1+\rho^{n-2})\varepsilon_{n-3}(y(n-3))\\ &\leq \left(\rho^{n-1}\prod_{i=1}^{n-1}(1+\rho^i)\right)\varepsilon_0(y(0))\\ &= a_n\eta(y)/\rho.\end{aligned}$$

Noting that $\varepsilon_n = \rho\varepsilon_{n-1}$ we have:

CLAIM 5.3: $|y(n)-y(n-1)|_1 \leq \varepsilon_n(y(n-1)) \leq a_n\eta(y)$ *for each* $n \in \mathbf{N}$.

Observe that

$$|y(n)-y|_1 \leq \sum_{j=1}^{n}|y(j)-y(j-1)|_1 \leq \sum_{j=1}^{n}a_j\eta(y) \leq \sum_{j=1}^{\infty}a_j\eta(y) \leq \eta(y)/3.$$

and so

$$|\eta(y(n))-\eta(y)| \leq \|y(n)-y\| \leq |y(n)-y|_1 \leq \eta(y)/3.$$

We now may conclude that

CLAIM 5.4: $\frac{2}{3}\eta(y) \leq \eta(y(n)) \leq \frac{4}{3}\eta(y)$ *for every* $n \in \mathbf{N}$.

Choose an $n \in \mathbf{N}$ such that $\eta(y) > 3/n$ and let $k \geq n$. So $\eta(y(k)) \geq \frac{2}{3}\eta(y) > 2/n$ and hence $y(k) \in G_n$. By property (2) of $g_{k+1}$ we have $g_{k+1}\circ f_k(x) = f_k(x)$. Consequently, $f_n(x) = f_{n+1}(x) = f_{n+2}(x) = \cdots$ for every $x$ in the open set

$$O_n = \{x \in X : \eta\circ H(x) > 3/n\}.$$

So if we put $f(x) = \lim_{i\to\infty} f_i(x)$, then $f\colon X \to X$ is a well defined function.

CLAIM 5.5: $f\colon X \to X$ *is a sign preserving homeomorphism with* $|H\circ f(x) - H(x)|_1 \leq \eta(H(x))/3$ *and* $|f(x)|_\infty = |x|_\infty$ *for every* $x \in X$.

*Proof:* Since $f|O_n = f_n|O_n$ and $\{O_i : i \in \mathbf{N}\}$ is an open covering of $X$, $f$ is continuous and open. Since $O_1 \subset O_2 \subset O_3 \subset \cdots$ we have that $f$ is one-to-one. Note that $f_n$ is sign preserving and sup norm preserving as a composition of $g_i$'s. Again, since $f|O_n = f_n|O_n$, $f$ has the same properties.

Let $z \in X$ and select an $n \in \mathbf{N}$ such that $\eta(H(z)) > 4/n$. Since $f_n$ is a homeomorphism we can find an $x \in X$ with $f_n(x) = z$. Then we have, with Claim 5.4,

$$\frac{4}{n} < \eta(H(z)) = \eta(H\circ f_n(x)) \leq \frac{4}{3}\eta(H(x)).$$

So we have $\eta(H(x)) > 3/n$ and $x \in O_n$. Hence $f(x) = f_n(x) = z$ and $f$ is a surjection.

For the inequality, let $n$ be such that $f(x) = f_n(x)$. Then

$$|H \circ f(x) - H(x)|_1 = |y(n) - y|_1 \leq \eta(y)/3 = \eta(H(x))/3.$$

Let $x' \in X$ be such that $H(x') = y = H(x)$. Define $y'(j) = H \circ f_j(x')$ for $j \in \omega$.

CLAIM 5.6: *For every $j \in \omega$, $y(j) = y'(j)$, and for every $j \in \omega$ and $m \in \mathbf{N}$ with $m - 1 \leq 1/\eta(y)$ and $m \leq j + 1$, we have*

$$|\pi_m(f_j(x) - f_j(x'))| \leq \left(1 + \frac{\eta(y)}{3}\right)^j \frac{\eta(y)}{3}.$$

*Proof:* We use induction with respect to $j$. If $j = 0$, then $f_0$ is the identity so the first part of the claim is void. The last part becomes $|x_1 - x'_1| \leq \eta(y)/3$, which is true since $x_1 = \pi_1(y) = x'_1$ by the definition of $H$.

Assume that the Claim is true for some $j - 1 \geq 0$. Then $y(j) = y'(j)$ follows from $y(j-1) = y'(j-1)$ with Property (7) for $g_j$.

For the inequality consider first the case $m \leq j$. We have that $m - 1 \leq 1/\eta(y)$. If we combine the assumptions $y(j-1) = y'(j-1)$ and

$$|\pi_m(f_{j-1}(x) - f_{j-1}(x'))| \leq \left(1 + \frac{\eta(y)}{3}\right)^{j-1} \eta(y)/3 \quad \text{for } m \leq j$$

with Property (8) of $g_j$ and Claim 5.3, then we obtain

$$\begin{aligned}
|\pi_m(f_j(x) - f_j(x'))| &= |\pi_m(g_j \circ f_{j-1}(x) - g_j \circ f_{j-1}(x'))| \\
&\leq (1 + \varepsilon(y(j-1)))|\pi_m(f_{j-1}(x) - f_{j-1}(x'))| \\
&\leq (1 + a_j\eta(y)) \left(1 + \frac{\eta(y)}{3}\right)^{j-1} \eta(y)/3 \\
&\leq \left(1 + \frac{\eta(y)}{3}\right)^j \eta(y)/3.
\end{aligned}$$

If $m = j + 1$, then we may conclude with Claim 5.4 that $\eta(y(j-1)) \leq \frac{4}{3}\eta(y) \leq 4/(3j)$ and hence that $y(j-1) \in F_j$. We have by Property (9) of $g_j$ and Claim 5.3 that

$$\begin{aligned}
|\pi_{j+1}(f_j(x) - f_j(x'))| &= |\pi_{j+1}(g_j \circ f_{j-1}(x)) - g_j \circ f_{j-1}(x'))| \\
&\leq \varepsilon_j(y(j-1)) \leq a_j\eta(y) \leq \eta(y)/3 \\
&\leq \left(1 + \frac{\eta(y)}{3}\right)^j \eta(y)/3.
\end{aligned}$$

The following claim shows that $f$ shrinks all fibres of $H$.

CLAIM 5.7: *If $H(x) = H(x')$ then $\|f(x) - f(x')\| \leq \delta(f(x))$.*

*Proof:* Let $n$ be such that $n - 1 \leq 1/\eta(y) < n$. Put $j = 3n$ and note that $\eta(y) > 3/j$, so $x, x' \in O_j$ and $f(x) = f_j(x)$ and $f(x') = f_j(x')$. According to Claim 5.6 we have, for each $m \leq n$,

$$|\pi_m(f(x) - f(x'))| = |\pi_m(f_j(x) - f_j(x'))| \leq \left(1 + \frac{\eta(y)}{3}\right)^j \frac{\eta(y)}{3}.$$

We have $j \leq \frac{3}{\eta(y)} + 3 \leq \frac{3}{\eta(y)}2$, so

$$\left(1 + \frac{\eta(y)}{3}\right)^j \leq \left(1 + \frac{\eta(y)}{3}\right)^{\frac{3}{\eta(y)}2} \leq e^2.$$

Consequently, for $m \leq n$, $|\pi_m(f(x) - f(x'))| \leq e^2\eta(y)/3$ and

$$\begin{aligned}\|(f(x) - f(x'))\| &\leq \max(\{1/n\} \cup \{|\pi_m(f(x) - f(x'))|/m : m < n\}) \\ &\leq \frac{e^2}{3}\eta(y) \leq \frac{e^2}{2}\eta(H \circ f(x)) \leq \frac{e^2}{8}\delta(f(x)) \leq \delta(f(x)),\end{aligned}$$

where we used $1/n < \eta(y)$ and $\eta(y) \leq \frac{3}{2}\eta(y(j)) = \frac{3}{2}\eta(H \circ f(x))$ (by Claim 5.4).

If we put $h = f^{-1}$, we see that the conditions for Theorem 4.1 have been met and Theorem 5.2 is proved.

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